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The Level 2 and 3 Modular Invariants for the Orthogonal Algebras

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Abstract. This paper finds for each affine algebra $B_r^{(1)}$ and $D_r^{(1)}$ all modular invariant 1-loop partition functions at level ≤ 3. Previously, only those at level 1 were classified. An extraordinary number of exceptionals appear at level 2 – indeed this is the motivation for this paper – and we find infinitely many new ones there. The only level 3 exceptionals occur for $B_2^{(1)} \cong C_2^{(1)}$ and $D_7^{(1)}$, and the latter appear to be new. The $B_{2,3}$ and $D_{7,3}$ exceptionals are cousins of the $\mathcal{E}_6$-exceptional and $\mathcal{E}_8$-exceptional, respectively, of $A_1^{(1)}$, while the level 2 exceptionals are related to the lattice invariants of affine $u(1)$.

1. Introduction

Over the past decade or so, much work has been directed towards one aspect of the classification of conformal field theories: the classification of modular invariant partition functions corresponding to the affine algebras. This classification question can be asked for any choice of algebra $X_r^{(1)}$ and level $k \in \mathbb{Z}_+ \stackrel{\text{def}}{=} \{0,1,2,\ldots\}$. An elaborate machinery has been developed, and we can be optimistic about our chances for the complete classification, at least when X_r is simple. Nevertheless, at the present time few of these classifications have been accomplished: the main successes are $X_r = A_1$, A_2, and $U_1 \oplus \cdots \oplus U_1$ (the quasi-rational unitary case should be interpreted rationally by equating representations with equal character), for all levels k [2,6,8]; and levels $k \leq 3$ for all A_r [7].

The result for A_1 falls into an A-D-E pattern [2]. Philippe Ruelle [15] discovered a connection between Jacobians of Fermat curves, and the A_2 classification. There is a natural relationship [8] between rational points on Grassmannians and the $U_1 \oplus \cdots \oplus U_1$ classification. Relations between these partition functions and subfactors in von Neumann algebra theory is discussed e.g. in [3]. For these reasons, as well of course for conformal field theory itself, it should certainly be of interest to make further efforts to obtain complete lists of these partition functions, and to understand better the curious relationships between those lists and other areas of mathematics and mathematical physics.

The finitely many characters χ_λ, $\lambda \in P_+$, of the integrable highest weight representations of $X_r^{(1)}$ at level k, carry a representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$ [12]. The matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which generate $\mathrm{SL}_2(\mathbb{Z})$, get sent to the Kac-Peterson matrices S and T, respectively. The entries $S_{\mu\nu}$ are related to values of Lie group characters at elements of finite order, while those of the diagonal matrix T are related to the eigenvalues of the quadradic Casimir. Our classification problem reduces to finding all matrices $M = (M_{\mu\nu})_{\mu,\nu \in P_+}$ which obey

(P1) $MS = SM$ and $MT = TM$
(P2) $M_{\mu\nu} \in \mathbb{Z}_+$ for all $\mu,\nu \in P_+$
(P3) $M_{00} = 1$

(P1) says that the corresponding partition function should be a modular invariant, (P2) applies because the entries of $M_{\mu\nu}$ count certain 'primary fields', while (P3) says that the vacuum is unique. Any such matrix M is called a *physical invariant*.

An approach has slowly evolved to handle these classifications. It breaks the problem into 2 parts:

(1) find all possible values of $M_{\mu 0}$ and $M_{0\mu}$, for all $\mu \in P_+$;
(2) find all physical invariants M, for each possible choice of values $M_{\mu 0}$ and $M_{0\mu}$ found in (1).

The point is that the values in (1) are severely constrained. What is found is that for any physical invariant of almost every choice of $X_{r,k}$ (i.e. algebra $X_r^{(1)}$ at level k),

$$M_{\mu 0} \neq 0 \quad \text{or} \quad M_{0\mu} \neq 0 \qquad \Longrightarrow \qquad \mu \in \mathcal{S}(0)\ , \tag{1.1}$$

where $\mathcal{S}$ is the group of symmetries of the Coxeter-Dynkin diagram of $X_r^{(1)}$ ($\mathcal{S}$ acts on P_+ by permuting the indices $0,1,2,\ldots,r$ of the weights λ). The orbit $\mathcal{S}(0)$ in (1.1) is the set of simple currents. For example, for $A_{1,k}$, all but two of the physical invariants obey (1.1) (those two are the so-called $\mathcal{E}_6$ and $\mathcal{E}_8$ exceptionals, at $k=10$ and 28 respectively). Any physical invariant obeying (1.1) is called an $\mathcal{ADE}_7$-type invariant, by analogy with the $A_{1,k}$ classification. All evidence points to the validity of the following conjecture:

for any fixed choice of simple algebra X_r,
all but finitely many physical invariants of $X_{r,k}$ will obey (1.1).

Hence an important – and recently completed [9,10] – component of the classification of all physical invariants for $X_{r,k}$, is to find all $\mathcal{ADE}_7$-type invariants. This can be thought of as the generic situation in (2). By contrast, little general work has been done on (1).

However this general programme breaks down for the orthogonal algebras at level 2. They behave completely uncharacteristically: part (1) has far too many solutions and (1.1) rarely holds. Thus $B_{r,2}$ and $D_{r,2}$ must be treated using novel arguments. This is the primary motivation for this paper.

On the other hand, $B_{r,3}$ and $D_{r,3}$ behave generically (all their physical invariants are of $\mathcal{ADE}_7$-type, except for $B_{2,3}$ and $D_{7,3}$), and we include their classification as an indication of more normal behaviour. There are only three level 3 exceptionals.

Level $k \leq 3$ physical invariants are also known for the algebra $A_r^{(1)}$ [7]. Low-level classifications essentially reduce to low-rank ones, because of a curious relation called *rank-level duality* (see e.g. [14]). For example, the Kac-Peterson matrices of $\widehat{so}_n$ level k and $\widehat{so}_k$ level n are related. In particular, we find that $B_{r,2}$ and $D_{r,2}$ are related to U_1 at levels $2r+1$ and $2r$, respectively, while $B_{r,3}$ and $D_{r,3}$ are related to $A_{1,4r+4}$ and $A_{1,4r+2}$.

2. The list of physical invariants

We begin this section with a quick review, and then we list all physical invariants for the orthogonal algebras at $k \leq 3$.

2.1. Definitions. A highest-weight $\lambda = \lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \cdots + \lambda_r\Lambda_r \in P_+$ in $B_{r,k}$ ($r \geq 3$) satisfies $k = \lambda_0 + \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{r-1} + \lambda_r$, while for $D_{r,k}$ ($r \geq 4$) $\lambda \in P_+$ satisfies $k = \lambda_0 + \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r$; in both cases the Λ_i are the fundamental weights and all $\lambda_i \in \mathbb{Z}_+$. For $B_{r,k}$ put $n = k+2r-1$, while for $D_{r,k}$ put $n = k+2r-2$. The Weyl vector is $\rho = \sum \Lambda_i$. Usually we will drop the redundant Λ_0 component of the weights. Note that $B_{2,k}$ is more properly called $C_{2,k}$, and so can, should, and will be ignored in the following.

The symmetries $\mathcal{S}$ of the Coxeter-Dynkin diagrams will play a major role. Those fixing the 0th node are called *conjugations*, while others – called *simple currents* – form an abelian group we will call $\mathcal{S}_{sc}$.

$B_{r,k}$ has no nontrivial conjugation. For any $D_{r,k}$, there is a conjugation C_1 interchanging $\lambda_{r-1} \leftrightarrow \lambda_r$. Put $C_0 = I$, the identity. When $r=4$, there are four additional conjugations $C_2,\ldots,C_5$ – these six C_i for $D_4^{(1)}$ correspond to the different permutations of its Dynkin labels $\lambda_1, \lambda_3, \lambda_4$.

$B_{r,k}$ has a simple current J_b of order 2, given by $J_b\lambda = (\lambda_1, \lambda_0, \lambda_2, \ldots, \lambda_r)$. There are three non-trivial simple currents for $D_{r,k}$, namely J_v, J_s and $J_c = J_v \circ J_s$, defined by $J_v\lambda = (\lambda_1, \lambda_0, \lambda_2, \ldots, \lambda_{r-2}, \lambda_r, \lambda_{r-1})$ and

$$J_s\lambda = \begin{cases} (\lambda_r, \lambda_{r-1}, \lambda_{r-2}, \ldots, \lambda_1, \lambda_0) & \text{if } r \text{ is even} \\ (\lambda_{r-1}, \lambda_r, \lambda_{r-2}, \ldots, \lambda_1, \lambda_0) & \text{if } r \text{ is odd} \end{cases} .$$

Write $\mathcal{J}_b = \{id., J_b\}$, $\mathcal{J}_v = \{id., J_v\}$, $\mathcal{J}_s = \{id., J_s\}$, and $\mathcal{J}_d = \{id., J_v, J_s, J_c\}$. By a *spinor* for $B_{r,k}$ or $D_{r,k}$, respectively, is meant any weight $\lambda \in P_+$ with λ_r or $\lambda_{r-1} + \lambda_r$ odd. Write $\mathcal{P}_b$ and $\mathcal{P}_v$ for the sets of nonspinors.

We say λ and μ are *M-coupled* if either $M_{\lambda\mu} \neq 0$ or $M_{\mu\lambda} \neq 0$. By a *positive invariant* we mean a matrix M commuting with the corresponding Kac-Peterson matrices S and T, with the additional property that each $M_{\mu\nu} \geq 0$. By a *physical invariant*, we mean a positive invariant with each $M_{\mu\nu} \in \mathbb{Z}$, and obeying (P3). By an *$\mathcal{ADE}_7$-type invariant*, we mean a physical invariant M satisfying (1.1). Finally, by an *automorphism invariant*, we mean a physical invariant obeying

$$M_{\lambda 0} = M_{0\lambda} = \delta_{\lambda,0} \ . \tag{2.1a}$$

Automorphism invariants are important examples of physical invariants. It turns out (see Lemma 3.1(c)) that any automorphism invariant will be a permutation matrix, i.e. there will be a permutation π of P_+ such that

$$M_{\lambda\mu} = \delta_{\mu,\pi\lambda} \ . \tag{2.1b}$$

Each conjugation C defines an automorphism invariant, which we will also denote by C, obtained by taking $\pi = C$ in (2.1b). Moreover, the matrix products $C\,M$ and $M\,C$ of C with any other physical invariant M will also be a physical invariant.

The primary reason for the importance of simple currents is: let $J \in \mathcal{S}_{sc}$, then

$$S_{J\mu,\nu} = \exp[2\pi \mathrm{i}\, Q_J(\nu)]\, S_{\mu\nu} \ , \tag{2.2a}$$

for some number $Q_J(\nu)$ [13,17]. $Q_{J_b}(\mu) = \lambda_r/2$, while $Q_{J_v}(\mu) = (\lambda_{r-1} + \lambda_r)/2$ and

$$Q_s(\lambda) = \sum_{j=1}^{r-2} j\lambda_j/2 - \frac{r-2}{4}\lambda_{r-1} - \frac{r}{4}\lambda_r \ .$$

The matrix T also behaves similarly under $\mathcal{S}_{sc}$:

$$\frac{(J\mu+\rho)^2 - (\mu+\rho)^2}{2n} \equiv \frac{R(J)\,(N-1)}{2N} - Q_J(\mu) \pmod 1 \ , \tag{2.2b}$$

where N is the order of J, and where $R(J)$ is some integer. $R(J_b) = R(J_v) = 2k$ and $R(J_s) = R(J_c) = N_s\,(N_s - 1)kr/4$, where N_s is the order of J_s: $N_s = 2$ or 4 depending on whether or not r is even.

From these equations, it is possible to find a sequence of physical (in fact $\mathcal{ADE}_7$-type) invariants, for each $J \in \mathcal{S}_{sc}$. In particular, define [17]

$$\mathcal{I}[J]_{\mu,\nu} = \sum_{\ell=1}^{N} \delta_{J^\ell\mu,\nu}\, \delta^1\big(Q_J(\mu) + \frac{\ell}{2N} R(J)\big) \ , \tag{2.3}$$

where $\delta^1(x) = 1$ if $x \in \mathbb{Z}$ and $= 0$ otherwise. For example, $\mathcal{I}[id.] = I$, the identity matrix. $\mathcal{I}[J]$ will be a physical invariant iff $R(J)$ is even.

Any physical invariant not constructable in these standard ways out of simple currents and conjugations is called an *exceptional invariant.*

2.2. The list of physical invariants for $B_{r,2}$. Here $n = 2r+1$. P_+ consists of precisely $r+4$ weights: 0, $J_b 0 = 2\Lambda_1$, Λ_r, $J_b\Lambda_r = \Lambda_1 + \Lambda_r$, $\gamma^i \stackrel{\text{def}}{=} \Lambda_i$ for $i < r$, and $\gamma^r \stackrel{\text{def}}{=} 2\Lambda_r$. Write γ^0 for the weight 0. To minimise subscripts, we will usually abbreviate 'J_b' to 'J'. Extra exceptionals exist when n is a perfect square: $\sqrt{n} \in \mathbb{Z}$. In that case $4|r$, and it is convenient to introduce the following notation: when $8|r$, write $\lambda^r \stackrel{\text{def}}{=} \Lambda_r$ and $\mu^r \stackrel{\text{def}}{=} J\Lambda_r$; otherwise write $\lambda^r \stackrel{\text{def}}{=} J\Lambda_r$ and $\mu^r \stackrel{\text{def}}{=} \Lambda_r$. Also write $\mathcal{C} = \{\gamma^a \neq 0 \,|\, \sqrt{n} \text{ divides } a\}$.

Define matrices $\mathcal{B}(d,\ell)$, $\mathcal{B}(d_1,\ell_1|d_2,\ell_2)$, $\mathcal{B}^i$, $\mathcal{B}^{ii}$, $\mathcal{B}^{iii}$, $\mathcal{B}^{iv}$ by:

$$\mathcal{B}(d,\ell)_{J^i\gamma^a,J^i\gamma^b} = \begin{cases} 2 & \text{if } d|a,\ d|b, \text{and both } a \neq 0,\ b \neq 0 \\ 0 & \text{if } n \not| da \text{ or } b \not\equiv \pm a\ell \pmod{d} \\ 1 & \text{otherwise} \end{cases}$$
$$\mathcal{B}(d,\ell)_{J^i\Lambda_r,J^i\Lambda_r} = 1$$

and all other entries are 0, where $a, b \in \{0, 1, \ldots, r\}$ and $i \in \{0, 1\}$;

$$\mathcal{B}(d_1,\ell_1|d_2,\ell_2) = \frac{1}{2}(\mathcal{B}(d_1,\ell_1) + \mathcal{B}(d_2,\ell_2))\,\mathcal{I}[J_b]$$
$$\mathcal{B}^i_{00} = \mathcal{B}^i_{0\gamma} = \mathcal{B}^i_{\gamma 0} = \mathcal{B}^i_{\gamma\gamma'} = \mathcal{B}^i_{\lambda^r\gamma} = \mathcal{B}^i_{\gamma\lambda^r} = \mathcal{B}^i_{\mu^r\mu^r} = \mathcal{B}^i_{\lambda^r,J0} = \mathcal{B}^i_{J0,\lambda^r} = 1$$
$$\mathcal{B}^{ii}_{00} = \mathcal{B}^{ii}_{0\gamma} = \mathcal{B}^{ii}_{\gamma 0} = \mathcal{B}^{ii}_{\gamma\gamma'} = \mathcal{B}^{ii}_{0\lambda^r} = \mathcal{B}^{ii}_{\lambda^r 0} = \mathcal{B}^{ii}_{\lambda^r\lambda^r} = \mathcal{B}^{ii}_{\lambda^r\gamma} = \mathcal{B}^{ii}_{\gamma\lambda^r} = 1$$

and all other entries are 0, where $\gamma, \gamma' \in \mathcal{C}$. Finally, $\mathcal{B}^{iii} = \mathcal{B}^i\,\mathcal{I}[J_b]$ and $\mathcal{B}^{iv} = \mathcal{I}[J_b]\,\mathcal{B}^i$.

In section 4.1 we will prove:

THEOREM 2.1. *Let M be a physical invariant of $B_{r,2}$. Then M equals one of the following:*

(a) $\mathcal{B}(d,\ell)$ for any divisor d of $n = 2r+1$ obeying $n|d^2$, and for any integer $0 \leq \ell < \frac{d^2}{2n}$ obeying $\ell^2 \equiv 1$ (mod $\frac{d^2}{n}$);

(b) $\mathcal{B}(d_1,\ell_1|d_2,\ell_2)$ for any divisors d_i of n obeying $n|d_i^2$, and for any integers $0 \leq \ell_i < \frac{d_i^2}{2n}$ obeying $\ell_i^2 \equiv 1$ (mod $\frac{d_i^2}{n}$);

(c) when n is a perfect square, there are 4 remaining physical invariants: $\mathcal{B}^i$, $\mathcal{B}^{ii}$, $\mathcal{B}^{iii}$, and $\mathcal{B}^{iv}$.

The only redundancy here is that $\mathcal{B}(d_1,\ell_1|d_2,\ell_2) = \mathcal{B}(d_2,\ell_2|d_1,\ell_1)$. There are a total of D distinct $\mathcal{B}(d,\ell)$'s, where D is the number of divisors $d' \leq \sqrt{n}$ of n. All but one

of these, namely $\mathcal{B}(n,1)=I$, are exceptional. There are precisely $D\,(D+1)/2$ distinct physical invariants in part (b), and all but one of them (namely $\mathcal{B}(n,1|n,1)=\mathcal{I}[J_b]$) are exceptional. When $\sqrt{n}\in\mathbb{Z}$, the physical invariants $\mathcal{B}^i$, $\mathcal{B}^{ii}$, $\mathcal{B}^{iii}$, $\mathcal{B}^{iv}$ are all distinct and exceptional. Of all $B_{r,2}$ physical invariants, only $\mathcal{B}^{iii}$ and $\mathcal{B}^{iv}$ are not symmetric matrices.

Most of these exceptionals are new. The $\mathcal{B}(d,\ell)$ for the special case $d=n$ are the exceptional automorphism invariants found in [4]. The $\mathcal{B}(d,\ell)$ for the special case $\ell=1$ are the exceptional invariants given in [16]. Multiplying these gives all invariants of type $\mathcal{B}(d,\ell)$.

For example, when $3\le r\le 12$, respectively, there are precisely 2, 9, 2, 2, 5, 2, 2, 5, 2, 9 physical invariants for $B_{r,2}$. All nine $B_{4,2}$ physical invariants can be found in the Appendix B of [18], and the correspondence between his notation and ours is: $Z_1=\mathcal{B}(9,1)$, $Z_2=\mathcal{B}(9,1|9,1)$, $Z_3=\mathcal{B}(3,1)$, $Z_4=\mathcal{B}^i$, $Z_5=\mathcal{B}^{ii}$, $Z_6=\mathcal{B}(3,1|3,1)$, $Z_7=\mathcal{B}^{iii}$, $Z_8=\mathcal{B}^{iv}$, and $Z_9=\mathcal{B}(3,1|9,1)$. Verstegen attributed this $B_{4,2}$ richness to the existence of conformal embeddings involving E_8, A_8 and D_8, and so was unaware that $B_{4,2}$ is merely the tip of an iceberg!

2.3. The list of physical invariants for $D_{r,2}$. There are $r+7$ weights: 0, $2\Lambda_1=J_v0$, $2\Lambda_{r-1}=J_c0$, $2\Lambda_r=J_s0$, Λ_r, $\Lambda_1+\Lambda_{r-1}=J_v\Lambda_r$, Λ_{r-1}, $\Lambda_1+\Lambda_r=J_v\Lambda_{r-1}$, $\lambda^i\stackrel{\text{def}}{=}\Lambda_i$ for $1\le i\le r-2$, and $\lambda^{r-1}\stackrel{\text{def}}{=}\Lambda_{r-1}+\Lambda_r$. Write $n=2r$. Write λ^0 for the weight 0 and λ^r for $2\Lambda_r$, and J for J_v. Additional exceptionals occur when r is an even perfect square, and in this case write $\mathcal{C}_j=\{\lambda^b\ne 0\,|\,2\frac{b}{\sqrt{r}}\equiv\pm j \pmod 8\}$ for $j=0,1,2,3,4$.

Define the matrices $\mathcal{D}(d,\ell)$, $\mathcal{D}(d_1,\ell_1|d_2,\ell_2)$, $\mathcal{D}^i$, $\mathcal{D}^{ii}$, and $\mathcal{D}^{iii}$, as follows:

$$\mathcal{D}(d,\ell)_{J^i\lambda^a,J^i\lambda^b}=\begin{cases}2 & \text{if } d|a,\ d|b,\ 2d|(a+b),\ \text{and } \{a,b\}\subseteq\{1,\ldots,r-1\}\\ 0 & \text{if } r\not|da \text{ or } b\not\equiv\pm a\ell \pmod{2d}\\ 1 & \text{otherwise}\end{cases}$$

$$\mathcal{D}(d,\ell)_{\lambda_s\lambda_s}=\begin{cases}1 & \text{if } 2d\not|r\\ 2 & \text{if } \lambda_s\in\{\Lambda_r,\Lambda_1+\Lambda_{r-1}\} \text{ and } 2d|r\\ 0 & \text{otherwise}\end{cases}$$

and all other entries are 0, where $a,b\in\{0,1,\ldots,r\}$, $i\in\{0,1\}$, and λ_s is any spinor;

$$\mathcal{D}(d_1,\ell_1|d_2,\ell_2)=\frac{1}{2}(\mathcal{D}(d_1,\ell_1)+\mathcal{D}(d_2,\ell_2))\,\mathcal{I}[J_v]$$
$$\begin{aligned}\mathcal{D}^i_{J^j\Lambda_r,J^j\Lambda_r}&=\mathcal{D}^i_{\Lambda_r\mu}=\mathcal{D}^i_{\mu\Lambda_r}=\mathcal{D}^i_{J\Lambda_r,\mu'}=\mathcal{D}^i_{\mu',J\Lambda_r}=\mathcal{D}^i_{\lambda\lambda'}=\mathcal{D}^i_{\gamma\gamma'}\\ &=\mathcal{D}^i_{J'0,J''0}=\mathcal{D}^i_{J'0,\nu}=\mathcal{D}^i_{\nu,J'0}=\mathcal{D}^i_{J'J0,\nu'}=\mathcal{D}^i_{\nu',J'J0}=1\end{aligned}$$

where $\lambda,\lambda'\in\mathcal{C}_0\cup\mathcal{C}_4$, $\mu\in\mathcal{C}_1$, $\mu'\in\mathcal{C}_3$, $\gamma,\gamma'\in\mathcal{C}_2$, $\nu\in\mathcal{C}_0$, $\nu'\in\mathcal{C}_4$, $J',J''\in\mathcal{J}_s$, and $j\in\{0,1\}$. All other entries equal 0. Finally, $\mathcal{D}^{ii}=\mathcal{D}^i\,\mathcal{I}[J_v]$ and $\mathcal{D}^{iii}=\mathcal{I}[J_v]\,\mathcal{D}^i$.

In section 4.2 we prove:

THEOREM 2.2. *Let M be a physical invariant of $D_{r,2}$. Then M equals one of the following, for arbitrary conjugations C_i, C_j:*

(a) $C_i\,\mathcal{D}(d,\ell)\,C_j$ for any divisor d of r obeying $r|d^2$, and for any integer $1\le\ell\le\frac{d^2}{r}$ obeying $\ell^2\equiv 1$ (mod $\frac{4d^2}{r}$);

(b) $\mathcal{D}(d_1,\ell_1|d_2,\ell_2)$ *for any divisors* d_i *of* r *obeying* $r|d_i^2$*, as well as the additional property that* $2d_1|r$ *iff* $2d_2|r$*, and for any integers* $1\le \ell_i\le \frac{d_i^2}{r}$ *obeying* $\ell_i^2\equiv 1 \pmod{\frac{4d_i^2}{r}}$*;*

(c) when r *is a perfect square and* $16|r$*, there are 8 other physical invariants:* $C_i\,\mathcal{D}^i\,C_j$*,* $C_i\,\mathcal{D}^{ii}$*, and* $\mathcal{D}^{iii}\,C_j$*.*

Take $C_i=I$ in (a) unless $2d|r$. To count these physical invariants, define $D\{m\}$ to be the number of divisors $d\le\sqrt{m}$ of m, and let 2^c be the exact power of 2 dividing r. Put $D=D\{r\}$ and $D_1=D-D_0$, where $D_0=D\{2r/2^c\}$ when 4 divides r, and $D_0=D$ otherwise. For $r\ne 4$, there are precisely $2D_0+4D_1$ distinct physical invariants in (a) and $D_0(D_0+1)/2+D_1(D_1+1)/2$ in (b). For $D_{4,2}$, there are a total of 16 physical invariants (namely, 6 of the form $\mathcal{D}(4,1)\,C_i$, 9 of the form $C_i\,\mathcal{D}(2,1)\,C_j$, and $\mathcal{D}(2,1|2,1)$).

Most of these physical invariants are new, and all but the conjugations of $\mathcal{D}(r,1)=I$ and $\mathcal{D}(r,1|r,1)=\mathcal{I}[J_v]$ (for all r), and $\mathcal{D}(r,r-1)=\mathcal{I}[J_s]$ and $\mathcal{D}(r,r-1|r,r-1)=\mathcal{I}[J_v]\,\mathcal{I}[J_s]$ (when $r\equiv 2 \pmod 4$), and $\mathcal{D}(\frac{r}{2},1)=\mathcal{I}[J_s]$ and $\mathcal{D}(\frac{r}{2},1|\frac{r}{2},1)=\mathcal{I}[J_v]\,\mathcal{I}[J_s]$ (when $4|r$) are exceptional. The exceptionals $\mathcal{D}(d,\ell)$ for the special cases $d=r$ and $\ell=1$, respectively, first appeared in [4] and [16].

For example, for $4\le r\le 16$, respectively, there are precisely 16, 3, 8, 3, 8, 7, 7, 3, 12, 3, 7, 7, and 22 physical invariants. Of these, 0, 0, 2, 0, 0, 4, 1, 0, 4, 0, 1, 4, and 14 are exceptional.

2.4. The orthogonal algebras at levels 3 and 1. Next consider $B_{r,3}$, $r\ge 3$. In section 5.1 we will show that there are no exceptional physical invariants (incidently, there is one for $C_{2,3}$ – recall $C_2\cong B_2$):

THEOREM 2.3. *The only physical invariants for* $B_{r,3}$ *are the identity matrix* $M=I$ *and the simple current invariant* $M=\mathcal{I}[J_b]$*.*

The physical invariant $\mathcal{I}[J_b]$ was first found in [1]. It is an automorphism invariant; the associated permutation is order 2. The $C_{2,3}$ exceptional is a conformal embedding.

Next look at $D_{r,3}$, for $r\ge 4$. In section 5.2 we will show that the only exceptional invariants occur at $D_{7,3}$.

THEOREM 2.4. *The complete list of* $D_{r,3}$ *physical invariants is:*

- C_i *and* $C_i\,\mathcal{I}[J_v]$*, valid for all* k*;*
- *in addition for* $r\equiv 4 \pmod 8$*,* $C_i\,M$ *for* $M=\mathcal{I}[J_s],\mathcal{I}[J_c],\mathcal{I}[J_s]\,\mathcal{I}[J_c]$ *and* $\mathcal{I}[J_c]\,\mathcal{I}[J_s]$*;*
- *in addition when 8 divides* r*,* $C_i\,M\,C_j$ *for* $M=\mathcal{I}[J_s]$ *and* $\mathcal{I}[J_v]\,\mathcal{I}[J_s]$*;*
- *finally, for* $D_{7,3}$*, there are the exceptionals* $C_i\,\mathcal{E}(D_{7,3})$*, where*

$$\mathcal{E}(D_{7,3})\stackrel{\text{def}}{=}\sum_{J\in\mathcal{J}_d}|\chi_{J0}+\chi_{J(\Lambda_1+\Lambda_5)}|^2+\sum_{J\in\mathcal{J}_d}|\chi_{J\Lambda_3}+\chi_{J(\Lambda_1+\Lambda_6+\Lambda_7)}|^2\ ,$$

where C_i *and* C_j *are any conjugations (i.e.* $i,j\in\{0,1\}$ *for* $r\ne 4$*, and* $i,j\in\{0,1,\dots,5\}$ *for* $r=4$*).*

$\mathcal{I}[J_v]$ and $\mathcal{I}[J_s]$ were first found in [1]. To our knowledge, $\mathcal{E}(D_{7,3})$ has never appeared before in the literature. $\mathcal{I}[J_v]$ is an automorphism invariant; the permutation will be order 2. When $r\equiv 4 \pmod 8$, $\mathcal{I}[J_s]$ and $\mathcal{I}[J_c]$ will also be order 2 automorphism invariants.

When 8 divides r, $\mathcal{I}[J_s]$ will be a direct sum of $r+2$ matrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $2r+4$ matrices (0).

For completeness, let us repeat the level 1 classification [5]. There are no exceptionals. For $B_{r,1}$ there is only the identity matrix I. For $D_{r,1}$, when 4 does not divide r, there are only 2 invariants: I and $C_1 = \mathcal{I}[J_v]$. When 4 divides r, there are a total of 6 invariants: C_i and $C_i\,\mathcal{I}[J_s]\,C_j$, for $i,j \in \{0,1\}$.

3. The Tools

The condition $TM = MT$ in (P1) is equivalent to the selection rule

$$M_{\lambda\mu} \neq 0 \Rightarrow (\lambda+\rho)^2 \equiv (\mu+\rho)^2 \pmod{2n}\ . \tag{3.1}$$

The other (P1) condition $SM = MS$ is much more subtle, as we will see.

The matrix S obeys an important symmetry. Its entries $S_{\lambda\mu}$ lie in some cyclotomic extension $\mathbb{Q}(\zeta_N)$ of $\mathbb{Q}$, where $\zeta_N = \exp[2\pi \mathrm{i}/N]$, so consider any Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \cong \mathbb{Z}_N^\times$. It will obey $\sigma\zeta_N = \zeta_N^\ell$ for some ℓ coprime to N. Then

$$\sigma(S_{\lambda\mu}) = \epsilon_\sigma(\lambda)\, S_{\lambda^\sigma\mu} \tag{3.2a}$$

for some permutation $\lambda \mapsto \lambda^\sigma$ of P_+, and some signs $\epsilon_\sigma : P_+ \to \{\pm 1\}$. Equation (3.2a) immediately implies [5,15]

$$M_{\lambda\mu} = \epsilon_\sigma(\lambda)\,\epsilon_\sigma(\mu)\, M_{\lambda^\sigma,\mu^\sigma} \tag{3.2b}$$

valid for any σ, any $\lambda,\mu \in P_+$, and any physical invariant M. Positivity (P2) then implies, for all σ, the selection rule

$$M_{\lambda\mu} \neq 0 \quad \Longrightarrow \quad \epsilon_\sigma(\lambda) = \epsilon_\sigma(\mu)\ . \tag{3.2c}$$

The selection rules (3.1) and (3.2c) are the most important ingredients (though there are others) of step (1) described in section 1. We will use (3.2c) in section 5.

For a positive invariant M, define

$$\mathcal{J}_L(M) = \{J \in \mathcal{S}_{sc} \,|\, M_{J0,0} \neq 0\}\ ; \tag{3.3a}$$

$$\mathcal{P}_L(M) = \{\lambda \in P_+ \,|\, \exists \mu \in P_+ \text{ such that } M_{\lambda\mu} \neq 0\}\ ; \tag{3.3b}$$

and define $\mathcal{J}_R(M)$ and $\mathcal{P}_R(M)$ similarly (using the other subscript of M). Call $\lambda \in P_+$ a *fixed-point* of $\mathcal{J} \subset \mathcal{S}_{sc}$ if the cardinalities satisfy $\|\mathcal{J}\lambda\| < \|\mathcal{J}\|$. For example, λ is a fixed-point of $\mathcal{J}_b$ iff $\lambda_0 = \lambda_1$. For any $\mathcal{J} \subset \mathcal{S}_{sc}$, define

$$\mathcal{P}(\mathcal{J}) \stackrel{\text{def}}{=} \{\lambda \in P_+ \,|\, Q_J(\lambda) \in \mathbb{Z} \quad \forall J \in \mathcal{J}\}\ . \tag{3.3c}$$

The following elementary lemma tells us how $\mathcal{J}_L(M)$ and $\mathcal{J}_R(M)$ affect the other entries of M.

LEMMA 3.1. [9] (a) *Let M be any physical invariant, and $J, J' \in \mathcal{S}_{sc}$. Then the following statements are equivalent:*

(i) $M_{J0,J'0} \neq 0$;
(ii) $M_{J0,J'0} = 1$;
(iii) *for any* $\lambda, \mu \in P_+$, *if* $M_{\lambda\mu} \neq 0$ *then* $Q_J(\lambda) \equiv Q_{J'}(\mu)$ *(mod 1)*;
(iv) $M_{J\lambda,J'\mu} = M_{\lambda\mu}$ *for all* $\lambda, \mu \in P_+$.

(b) *Let M be any positive invariant satisfying*

$$M_{0\mu} = \sum_{J \in \mathcal{J}_R} \delta_{\mu,J0}, \qquad M_{\lambda 0} = \sum_{J \in \mathcal{J}_L} \delta_{\lambda,J0} \ , \tag{3.4}$$

for some $\mathcal{J}_L, \mathcal{J}_R \subseteq \mathcal{S}_{sc}$. Then

(i) $\mathcal{J}_L$ *and* $\mathcal{J}_R$ *are groups and* $\|\mathcal{J}_L\| = \|\mathcal{J}_R\|$.
(ii) $\mathcal{P}_L(M) = \mathcal{P}(\mathcal{J}_L)$ *and* $\mathcal{P}_R(M) = \mathcal{P}(\mathcal{J}_R)$.

(c) *Let M be any physical invariant satisfying* (3.4), *for $\mathcal{J}_L = \mathcal{J}_R = \mathcal{J}$. Suppose no $\lambda \in \mathcal{P}(\mathcal{J})$ is a fixed-point of $\mathcal{J}$. Then there is a permutation π of the $\mathcal{J}$-orbits $\mathcal{P}(\mathcal{J})/\mathcal{J}$ such that*

$$M_{\lambda\mu} = \begin{cases} 1 & \text{if } \mu \in \pi(\mathcal{J}\lambda) \\ 0 & \text{otherwise} \end{cases} \ , \tag{3.5a}$$

$$S_{\lambda\mu} = S_{\pi\lambda,\pi\mu} \ , \tag{3.5b}$$

valid for all $\lambda, \mu \in \mathcal{P}(\mathcal{J})$ (the other entries of M all vanish).

For example, any $\mathcal{ADE}_7$-type invariant M will obey (3.4) with $\mathcal{J}_L = \mathcal{J}_L(M)$ and $\mathcal{J}_R = \mathcal{J}_R(M)$. Note that a special case of Lemma 3.1(c) is that automorphism invariants (2.1a) are permutation matrices (2.1b). Lemma 3.1 is a special case of Lemmas 3.1 and 3.2(b) in [9]. In particular, an analogue of 3.1(c) holds even if $\mathcal{J}_L \neq \mathcal{J}_R$ and if there are fixed-points, but this simpler case is sufficient for our purposes.

4. The level 2 physical invariant classifications

4.1. The $B_{r,2}$ physical invariant classification. The orthogonal algebras at level 2 behave essentially as if they were rank 1, so the "brute-force" approach of Cappelli-Itzykson-Zuber [2] can be modified to yield an efficient attack on the problem. In particular, "unfolding" $\mathcal{P}_b$ and $\mathcal{P}_v$ puts us into the familiar terrain of U_1 at level $n(= 2r+1$ or $2r$, respectively). We can explicitly find an integral basis for its commutant, and this basis quickly solves our classification problem. Three differences between our approach and that of [2] are that: (i) for $B_{r,2}$, our commutant is for the subgroup $\Gamma_\theta = \langle S, T^2 \rangle$ and not the full modular group $\Gamma = \langle S, T \rangle$; (ii) folding here preserves positivity; (iii) our approach applies directly only to the subsets $\mathcal{P}_b$ and $\mathcal{P}_v$ of $P_+(B_{r,2})$ and $P_+(D_{r,2})$.

We consider first $B_{r,2}$. Recall the notation introduced at the beginning of section 2.2. All S- and T-matrix entries for $B_{r,2}$ are obtained from

$$S_{00} = \frac{1}{2} S_{0\gamma^a} = \frac{1}{\sqrt{n}} S_{0,J^j\Lambda_r} = \frac{1}{2\sqrt{n}} \ , \tag{4.1a}$$

$$S_{\Lambda_r\Lambda_r} = S_{J\Lambda_r,J\Lambda_r} = -S_{\Lambda_r,J\Lambda_r} = \frac{1}{2}\ , \tag{4.1b}$$

$$S_{\gamma^a\gamma^b} = \frac{2}{\sqrt{n}}\cos\frac{2\pi ab}{n}\ , \tag{4.1c}$$

$$S_{\Lambda_r\gamma^a} = S_{J\Lambda_r,\gamma^a} = 0\ , \tag{4.1d}$$

$$(\gamma^a+\rho)^2 = \rho^2 + na - a^2\ , \tag{4.1e}$$

$$(J^j\Lambda_r+\rho)^2 = \rho^2 + jn + \frac{r}{4}n\ , \tag{4.1f}$$

for each $a, b \in \{1, \ldots, r\}$ and $j \in \{0, 1\}$. The missing values $S_{J0,*}$ can be obtained from (4.1a) by (2.2a), and all other S entries come from S being symmetric. These expressions (4.1a) - (4.1d) immediately follow from the calculations leading to rank-level duality, and they can also be found in [13]. Note from (4.1e) that $M_{\gamma^a\gamma^b} \neq 0$ requires $a^2 \equiv b^2 \pmod{n}$. A curiousity of this S-matrix is that it is essentially the character table for the dihedral group D_n.

The remainder of this subsection is devoted to the proof of Thm. 2.1. We will accomplish this by "unfolding". In particular, define $\widetilde{\mathcal{P}}_n = \mathbb{Z}/n\mathbb{Z}$, and

$$\tilde{S}_{ab} = \frac{1}{\sqrt{n}}\exp[2\pi \mathrm{i} ab/n] \tag{4.2a}$$

$$\tilde{T}^2_{ab} = \delta_{a,b}\ \exp[2\pi\mathrm{i}a^2/n] \tag{4.2b}$$

for all $a, b \in \widetilde{\mathcal{P}}_n$. Then it is easy to see directly that there is a bijection between all physical invariants M of $B_{r,2}$ with $\mathcal{J}_L(M) = \mathcal{J}_R(M) = \mathcal{J}_b$, and all nonnegative integral matrices $\widetilde{M}$ obeying:

$(\widetilde{P1})$ $\widetilde{M}$ commutes with $\tilde{S}$ and $\tilde{T}^2$;

$(\widetilde{P2})$ $\widetilde{M}_{\pm a,b} = \widetilde{M}_{a,\pm b} = \widetilde{M}_{ab}$, for any $a, b \in \tilde{\mathcal{P}}_n$;

$(\widetilde{P3})$ $\widetilde{M}_{00} = 4$, and $\widetilde{M}_{ab} \in 2\mathbb{Z}$ if either $a = 0$ or $b = 0$.

Precisely, the bijection is given by

$$M_{\gamma^a\gamma^b} = \widetilde{M}_{ab}\begin{cases} 1 & \text{if both } a \neq 0 \text{ and } b \neq 0 \\ \frac{1}{4} & \text{if } a = b = 0 \\ \frac{1}{2} & \text{otherwise}\end{cases} \tag{4.2c}$$

Lemma 4.1. (a) *A basis for the vector space $\tilde{\mathcal{V}}_n$ of all matrices commuting with $\tilde{S}$ and $\tilde{T}^2$, is provided by the set of matrices $\tilde{\mathcal{B}}(d, \ell)$:*

$$\tilde{\mathcal{B}}(d,\ell)_{a,b} = \begin{cases} 1 & \text{if } n|da, \text{ and } b \equiv a\ell \pmod{d} \\ 0 & \text{otherwise}\end{cases} \tag{4.3}$$

where $d|n$, $n|d^2$, and $1 \leq \ell \leq \frac{d^2}{n}$ obeys $\ell^2 \equiv 1 \pmod{\frac{d^2}{n}}$.

(b) *Any integral positive invariant M of $B_{r,2}$ with $M_{J0,0} = M_{0,J0} = M_{00}$ can be written as a sum of various $\mathcal{B}(d_1, \ell_1|d_2, \ell_2)$.*

There is a natural geometric interpretation of the $\tilde{\mathcal{B}}(d,\ell)$ in terms of self-dual lattices, and indeed that interpretation is the most convenient description of the physical invariant classification for $U_1\oplus\cdots\oplus U_1$ (see [8]).

Proof of Lemma 4.1. It is straightforward to verify that the matrices $\tilde{\mathcal{B}}(d,\ell)$ commute with $\tilde{S}$ and $\tilde{T}^2$. Also, they are all distinct and can be counted, and we find their number equals the number of divisors of n.

These $\tilde{\mathcal{B}}(d,\ell)$ possess an important property: given any matrix $\tilde{\mathcal{B}}(d,\ell)$, we can find an index (i,j) such that $\tilde{\mathcal{B}}(d',\ell')_{i,j}\neq 0$ iff $\tilde{\mathcal{B}}(d',\ell')=\tilde{\mathcal{B}}(d,\ell)$. To see this, choose $\ell_0\equiv\ell$ (mod d^2/n) so that $1\leq\ell_0\leq n$ and $\ell_0^2\equiv 1$ (mod n) – there may be more than one ℓ_0 corresponding to a given ℓ. Then $\mathcal{B}(d,\ell)=\mathcal{B}(d,\ell_0)$. The reason ℓ_0 is more convenient than ℓ is that the ℓ_0^2 condition means (i) $\ell_0\equiv\pm1$ (mod p^a) whenever p^a divides n, and (ii) together these signs uniquely determine ℓ_0. Let m be any integer for which both m and $2\ell_0+m\frac{d^2}{n}$ are coprime to n (m exists, by the Chinese Remainder Theorem). Then we have

$$\tilde{\mathcal{B}}(d',\ell')_{\frac{n}{d},\ell_0\frac{n}{d}+md}=\begin{cases}1 & \text{if } d=d' \text{ and } \ell_0\equiv\ell' \text{ (mod } \frac{d^2}{n})\\ 0 & \text{otherwise}\end{cases}\quad, \tag{4.4}$$

for any $\tilde{\mathcal{B}}(d',\ell')$ (Proof: The top equation is clear from (4.3); to see the bottom equation, suppose $\tilde{\mathcal{B}}_{\frac{n}{d},\ell_0\frac{n}{d}+md}=1$. Then (4.3) says d must divide d'. If p^a divides dd'/n for $a>0$, then $p^a|n$ so p will be coprime to m and $2\ell_0+md^2/n$, and hence by (4.3) and (i), $\ell_0\equiv\ell_0'$ (mod p^a) and $p^a|\frac{d^2}{n}$. This means by (ii) that $\ell_0\equiv\ell_0'$ (mod $\frac{dd'}{n}$) and $\frac{dd'}{n}$ divides $\frac{d^2}{n}$. Hence $d=d'$ and $\ell=\ell'$.).

An immediate consequence of (4.4) is that the $\tilde{\mathcal{B}}(d,\ell)$ are linearly independent. Thus to conclude the proof of part (a), it suffices to show that dim $\tilde{\mathcal{V}}$ is at most the number of divisors of n.

We will now follow the proof of Thm. 2 in [5] (which in turn is based on the argument of [2]). For each $u,u'\in\widetilde{\mathcal{P}}_{2n}=\mathbb{Z}/2n\mathbb{Z}$, define an $n\times n$ matrix $\{u,u'\}$ by

$$\{u,u'\}_{a,b}=\delta^{(n)}_{a,u+b}\exp[\pi\mathrm{i}\,(u+2b)\,u'/n] \tag{4.5a}$$

for all $a,b\in\widetilde{\mathcal{P}}_n$, where $\delta^{(n)}_{x,y}$ equals zero unless n divides $x-y$, when it equals 1. $\mathrm{SL}_2(\mathbb{Z})$ acts on the right by $\{u,u'\}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\{au+cu',bu+du'\}$. Note that

$$\tilde{T}^2\{u,u'\}\tilde{T}^{2*}=\{u,u'\}\begin{pmatrix}1&2\\0&1\end{pmatrix} \tag{4.5b}$$

$$\tilde{S}\{u,u'\}\tilde{S}^{*}=\{u,u'\}\begin{pmatrix}0&1\\-1&0\end{pmatrix} \tag{4.5c}$$

The index of the subgroup $\Gamma_\theta=\langle\begin{pmatrix}0&1\\-1&0\end{pmatrix},\begin{pmatrix}1&2\\0&1\end{pmatrix}\rangle$ in the modular group $\Gamma=\mathrm{SL}_2(\mathbb{Z})$ is 3, with left cosets Γ_θ, $\begin{pmatrix}1&1\\0&1\end{pmatrix}\Gamma_\theta$, and $\begin{pmatrix}1&0\\1&1\end{pmatrix}\Gamma_\theta$. Certainly $\tilde{\mathcal{V}}$ is spanned by the orbit

sums $N_\theta(u,u')\stackrel{\rm def}{=}\sum_{g\in\Gamma_\theta}\{u,u'\}g$, but it is more convenient to work over the orbit sums $N(u,u')\stackrel{\rm def}{=}\sum_{g\in\Gamma}\{u,u'\}g=(1+(-1)^u+(-1)^{u'})\,N_\theta(u,u')$. To see this relation, note that $\begin{pmatrix}1&1\\0&1\end{pmatrix}\equiv\begin{pmatrix}1&2\\0&1\end{pmatrix}^{\frac{n+1}{2}}$ and

$$\begin{pmatrix}1&0\\1&1\end{pmatrix}\equiv(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}^{-1})^{\frac{n-1}{2}}\ ,$$

both taken (mod n). Now for any $u_1,u_2\in\widetilde{\mathcal{P}}_{2n}$, $N(u_1,u_2)=N(d,0)$ where $d=\gcd(n,u_1,u_2)$.

Thus the number of linearly independent $N(u_1,u_2)$ (and hence the dimension of $\tilde{\mathcal{V}}$) is at most the number of divisors of n, and we are done part (a).

Part (b) is an immediate consequence of Lemma 3.1(b), part (a) and (4.4): unfolding M into $\widetilde{M}$ via (4.2c), we can write $\widetilde{M}$ as a sum of $\widetilde{M}_{00}=4M_{00}$ matrices $\tilde{\mathcal{B}}(d_i,\ell_i)$'s; by $(\widetilde{P2})$, these can be paired up so that $d_i=d_j$ and $\ell_i=-\ell_j$; finally, arbitrarily pair up these pairs (possible, since 4 divides $4M_{00}$) and refold, and we obtain $\mathcal{B}(d_i,\ell_i|d_i',\ell_i')$. ■

Lemma 4.1 immediately implies both the $(\widetilde{P1})$ - $(\widetilde{P3})$ classification and, more importantly, the classification of all physical invariants M of $B_{r,2}$ with $\mathcal{J}_{L,R}(M)=\mathcal{J}_b$. These are precisely the $\mathcal{B}(d_1,\ell_1|d_2,\ell_2)$ collected in Thm. 2.1(b).

Next, consider the possibility that $\mathcal{J}_L(M)=\mathcal{J}_R(M)=\{id.\}$, but $M_{\lambda_s\lambda_n}=M_{\lambda_n\lambda_s}=0$ for any *spinor* $\lambda_s\in\mathcal{J}_b\Lambda_r$ and any *nonspinor* $\lambda_n\in\mathcal{P}_b$. By Lemma 3.1(a), this means $M_{J0,J0}=1$, and also $M_{J\Lambda_r,J\Lambda_r}=M_{\Lambda_r\Lambda_r}$, $M_{0\gamma^a}=M_{J0,\gamma^a}$, etc. Of course (2.2b) and (3.1) tell us $M_{J\Lambda_r,\Lambda_r}=M_{\Lambda_r,J\Lambda_r}=0$. Computing $SM=MS$ at $(\Lambda_r,0)$ gives us $M_{\Lambda_r\Lambda_r}=1$, fixing all entries of M involving spinors. Finally, note that $M'=M\,\mathcal{I}[J]$ is also a physical invariant of $B_{r,2}$ but with $\mathcal{J}_L(M')=\mathcal{J}_R(M')=\mathcal{J}_b$. Hence $M'=\mathcal{B}(d_1,\ell_1|d_2,\ell_2)$ for some d_i,ℓ_i. Since every $M'_{\gamma^a\gamma^b}$ will be even (for $a,b>0$), we must have $d_1=d_2$ and $\ell_1=\ell_2$. This fixes all entries of M, and we get $M=\mathcal{B}(d_1,\ell_1)$.

From the calculations (4.1a) and the Galois selection rule (3.2c), this concludes the proof whenever n is not a perfect square. So consider now $\sqrt{n}\in\mathbb{Z}$. We may assume $M_{J0,J0}=0$ and without loss of generality that $M_{J0,0}=0$. Then T-invariance (3.1) and (4.1e),(4.1f) say that for any $\nu\neq\mu^r$,

$$M_{\mu^r\nu}=M_{\nu\mu^r}=0\ . \tag{4.6a}$$

$MS=SM$ evaluated at $(0,\mu^r)$, $(\mu^r,0)$ and $(J0,\mu^r)$, (μ^r,γ^a), and (μ^r,λ^r) gives us

$$\begin{aligned}M_{\mu^r\mu^r}&=1-M_{0,J0}-M_{0\lambda^r}=1-M_{\lambda^r0}=M_{J0,\lambda^r} &(4.6b)\\ M_{\lambda^r\gamma^a}&=M_{0\gamma^a}-M_{J0,\gamma^a} &(4.6c)\\ M_{\lambda^r\lambda^r}&=M_{0\lambda^r} &(4.6d)\end{aligned}$$

for all $a>0$. Note that (3.1) and (4.1e),(4.1f) say $M_{\lambda^r\gamma^a}=M_{\gamma^a\lambda^r}=0$ unless $\sqrt{n}$ divides a. So comparing $MS=SM$ at $(\lambda^r,0)$ and (λ^r,γ) for any $\gamma\in\mathcal{C}$, and using (4.6c), gives

$$M_{\lambda^r\gamma}=1+M_{\lambda^r0}-M_{\lambda^r\lambda^r}\ . \tag{4.6e}$$

By (4.6b), there are three possibilities:

(i) $M_{0,J0}=M_{\lambda^r0}=1$ and $M_{\mu^r\mu^r}=M_{*,\lambda^r}=0$;

(ii) $M_{0,J0}=M_{\mu^r\mu^r}=M_{J0,\lambda^r}=0$ and $M_{0\lambda^r}=M_{\lambda^r0}=1$;

(iii) $M_{0,J0}=M_{0\lambda^r}=M_{\lambda^r0}=0$ and $M_{\mu^r\mu^r}=M_{J0,\lambda^r}=1$.

In possibility (i), consider the product $\mathcal{I}[J]\,M$: it will be a physical invariant and so by Lemma 4.1(b) will equal some $\mathcal{B}(d,\ell|d,\ell)$. Hence for each γ^a,

$$M_{0\gamma^a}+M_{J0,\gamma^a}=(\mathcal{I}[J]\,M)_{0\gamma^a}=(\mathcal{I}[J]\,M)_{\gamma^a0}=2\,M_{\gamma^a0}$$

will equal either 0 or 2. Together with (4.6c) and (4.6e), we get $M_{\lambda^r\gamma}=M_{0\gamma}=2$, $M_{\gamma0}=1$ and $M_{J0,\gamma}=0$ for $\gamma\in\mathcal{C}$. Hence we must have $\mathcal{I}[J]\,M=\mathcal{B}(\sqrt{n},1|\sqrt{n},1)$, and we can read off the remaining entries: $M_{\gamma\gamma'}=2$ or 0, depending on whether or not both γ and γ' lie in $\mathcal{C}$. We thus obtain $M=\mathcal{B}^{iii}$.

For possibility (ii), use $M\,\mathcal{I}[J]=\mathcal{B}^{iii}$ and $\mathcal{I}[J]\,M=(\mathcal{B}^{iii})^T$, in order to show $M=\mathcal{B}^{ii}$. Similarly, for possibility (iii) we find $M=\mathcal{B}^{i}$.

4.2. The $D_{r,2}$ physical invariant classification. Next we consider $D_{r,2}$. The argument is very analogous to the $B_{r,2}$ one. Recall the notation introduced at the beginning of section 2.3. By the usual calculations (e.g. writing $(S_{\Lambda_r\Lambda_r}\pm S_{\Lambda_r\Lambda_{r-1}})/S_{0\Lambda_r}$ as a product of sines/cosines) we get

$$S_{00}=\frac{1}{\sqrt{r}}S_{0\Lambda_r}=\frac{1}{2}S_{0\lambda^a}=\frac{1}{2\sqrt{n}},\tag{4.7a}$$

$$S_{\lambda^a\lambda^b}=\frac{2}{\sqrt{n}}\cos(\pi\frac{ab}{r}),\tag{4.7b}$$

$$S_{\lambda^a\Lambda_r}=S_{\lambda^a\Lambda_{r-1}}=0\tag{4.7c}$$

$$S_{\Lambda_r\Lambda_r}=S_{\Lambda_{r-1}\Lambda_{r-1}}=\frac{1}{4}(1+(-\mathrm{i})^r),\tag{4.7d}$$

$$S_{\Lambda_r\Lambda_{r-1}}=\frac{1}{4}(1-(-\mathrm{i})^r),\tag{4.7e}$$

$$(\lambda^c+\rho)^2=\rho^2+2rc-c^2,\tag{4.7f}$$

$$(J^j\Lambda_r+\rho)^2=(J^j\Lambda_{r-1}+\rho)^2=\rho^2+jn+\frac{r^2}{2}-\frac{r}{4},\tag{4.7g}$$

where $a,b\in\{1,2,\ldots,r-1\}$, $c\in\{0,1,\ldots,r\}$ and $j\in\{0,1\}$. The remaining entries of S are given by (2.2a) and $S=S^T$. Again we have the curious relation between this matrix S and the dihedral group D_{2r}.

We will "unfold" $D_{r,2}$ as we did $B_{r,2}$. Namely, define $\widetilde{\mathcal{P}}_n=\mathbb{Z}/n\mathbb{Z}$ as before, and write

$$\tilde{S}_{ab}=\frac{1}{\sqrt{n}}\exp[2\pi\mathrm{i}ab/n]$$
$$\tilde{T}_{ab}=\delta_{a,b}\,\exp[\pi\mathrm{i}a^2/n]$$

for all $a,b\in\widetilde{\mathcal{P}}_n$. The difference here is that we are able to define $\tilde{T}$ rather than merely $\tilde{T}^2$ – this simplifies the arguments. It is easy to see directly that there is a bijection between

all physical invariants M of $D_{r,2}$ with both $J_v \in \mathcal{J}_L(M)$, $J_v \in \mathcal{J}_R(M)$, and all nonnegative integral $n \times n$ matrices $\widetilde{M}$ obeying:

$(\widetilde{P1})$ $\widetilde{M}$ commutes with $\tilde{S}$ and $\tilde{T}$;

$(\widetilde{P2})$ $\widetilde{M}_{\pm a,b} = \widetilde{M}_{a,\pm b} = \widetilde{M}_{ab}$, for any $a, b \in \widetilde{\mathcal{P}}_n$;

$(\widetilde{P3})$ $\widetilde{M}_{00} = 4$, $\widetilde{M}_{ab} \in 4\mathbb{Z}$ for $a, b \in \{0, r\}$, $\widetilde{M}_{ab} \in 2\mathbb{Z}$ if either $a \in \{0, r\}$ or $b \in \{0, r\}$.

In fact the bijection is given by

$$M_{\lambda^a \lambda^b} = \widetilde{M}_{ab} \begin{cases} 1 & \text{if both } a, b \notin \{0, r\} \\ \frac{1}{4} & \text{if both } a, b \in \{0, r\} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

LEMMA 4.2. (a) *A basis for the vector space $\tilde{\mathcal{V}}_n$ of all matrices commuting with $\tilde{S}$ and $\tilde{T}$, is provided by the set of all matrices $\tilde{\mathcal{B}}(d, \ell)$ given by* (4.3)*, where here $d|n$, $2n|d^2$, and $1 \leq \ell \leq \frac{d^2}{n}$ obeys $\ell^2 \equiv 1$ (mod $\frac{2d^2}{n}$).*

(b) *Any positive integral invariant M of $D_{r,2}$ with $M_{J0,0} = M_{0,J0} = M_{00}$, can be written as a sum of various $\mathcal{D}(d_1, \ell_1|d_2, \ell_2)$.*

Lemma 4.2(a) is a special case of Thm.2 in [5] and follows from a simplified version of our proof of Lemma 4.1. Counting the dimension of $\tilde{\mathcal{V}}_n$ as in Lemma 4.1(a), we see that it equals the number of divisors of $n/2 = r$.

Lemma 4.2(b) immediately gives us the classification of all $D_{r,2}$ physical invariants with $J_v \in \mathcal{J}_{L,R}(M)$: they are the $\mathcal{D}(d_1, \ell_1|d_2, \ell_2)$ given in Thm. 2.2(b).

Next, suppose both $J \notin \mathcal{J}_L(M)$ and $J \notin \mathcal{J}_R(M)$, but $M_{\lambda_s \lambda_n} = M_{\lambda_n \lambda_s} = 0$ for any spinor $\lambda_s \in \{C_1^i J^j \Lambda_r\}$, and any non-spinor $\lambda_n \in \mathcal{P}_v$. By Lemma 3.1(a), this means $M_{J0,J0} = 1$. Comparing $SM = MS$ at $(\lambda_s, J'0)$ for each choice of $\lambda_s \in \{\Lambda_{r-1}, \Lambda_r\}$ and $J' \in \mathcal{J}_s$, we find that either:

- (replacing M if necessary by its conjugation $M\,C_1$) $M_{J_s0,J_s0} = M_{\Lambda_r\Lambda_r} = M_{\Lambda_{r-1}\Lambda_{r-1}} = 1$, $M_{J_s0,J_c0} = M_{\Lambda_r\Lambda_{r-1}} = M_{\Lambda_{r-1}\Lambda_r} = 0$, and $\mathcal{J}_{L,R}(M) = \{id.\}$;
- (replacing M with some $C_1^i\,M\,C_1^j$) $M_{\Lambda_r\Lambda_r} = 2$, $\Lambda_{r-1} \notin \mathcal{P}_{L,R}(M)$, and $\mathcal{J}_{L,R}(M) = \mathcal{J}_s$; or
- (replacing M with some $C_1^i\,M\,C_1^j$ and if necessary transposing) $\mathcal{J}_L(M) = \mathcal{J}_s$ and $\mathcal{J}_R(M) = \{id.\}$.

The third possibility is eliminated by evaluating $SM = MS$ at (0,0): the left side is an even multiple of S_{00}, while the right side is an odd multiple of S_{00}. The remaining entries of M in the first two cases are fixed by Lemma 4.2(b) and the projection $M \mapsto M\,\mathcal{I}[J]$. We find that in either case, M (appropriately conjugated) equals one of the $\mathcal{D}(d, \ell)$ of Thm.2.2(a).

This concludes the classification of all physical invariants whenever r is not a perfect square, by the Galois argument (3.2c) applied to (4.7a), or whenever 4 does not divide r, by T-invariance (3.1). So consider now r a perfect square, $4|r$, $M_{J0,J0} = 0$ and without loss of generality $M_{J0,0} = 0$. T-invariance also says that spinors cannot couple to $\mathcal{J}_d0$. Recall the definition of $\mathcal{C}_j$ given in subsection 2.3. Then T-invariance says that λ^a can couple to $\mathcal{J}_d0$ only for $\lambda^a \in \mathcal{C}_0 \cup \mathcal{C}_4$, λ^b can couple to Λ_{r-1} or Λ_r only for $\lambda^b \in \mathcal{C}_1$, and λ^c can couple to $J\Lambda_{r-1}$ or $J\Lambda_r$ only for $\lambda^c \in \mathcal{C}_3$.

Lemma 4.2(b) tells us that either $\mathcal{I}[J]\,M$ (if $M_{0,J0}=1$) or $\mathcal{I}[J]\,M\,\mathcal{I}[J]$ (if $M_{0,J0}=0$) equals $\mathcal{D}(d_1,\ell_1|d_2,\ell_2)$ for d_i,ℓ_i as in Thm. 2.2(b). Because the $(2d_i,0)$ and $(\frac{r}{d_i},\ell_i\frac{r}{d_i})$ entries of the product will be even, we find that $d_1=d_2\stackrel{\rm def}{=}d$ and $\ell_1=\ell_2\stackrel{\rm def}{=}\ell$.

For any choice of $\lambda_s\in\{\Lambda_{r-1},\Lambda_r\}$, $MS=SM$ evaluated at (λ_s,λ) gives

$$\frac{2}{\sqrt{r}}\sum_{\nu\in\mathcal{C}_1}M_{\lambda_s\nu}=M_{\lambda_s\lambda'}=\frac{\pm1}{2}(M_{0\lambda_\pm}-M_{J0,\lambda_\pm}+s(\lambda_s)M_{J_s0,\lambda_\pm}-s(\lambda_s)M_{J_c0,\lambda_\pm})$$

where $\lambda_+\in\mathcal{C}_0$, $\lambda_-\in\mathcal{C}_4$, $\lambda'\in\mathcal{C}_1$, and $s(\lambda_s)=\exp[2\pi\mathrm{i}\,Q_s(\lambda_s)]\in\{\pm1\}$. The first inequality says that, for fixed $\lambda_s\in\{\Lambda_{r-1},\Lambda_r\}$, $M_{\lambda_s\lambda'}$ is independent of $\lambda'\in\mathcal{C}_1$ (call this value $\mathcal{M}_L(\lambda_s)$). Evaluating $SM=MS$ at (λ_s,μ) for $\mu\in\mathcal{C}_3$ shows $M_{J\lambda_s,\mu}$ is also constant and equals $\mathcal{M}_L(\lambda_s)$.

Comparing $\lambda_s\in\{\Lambda_{r-1},\Lambda_r\}$ in the second equality, we get $M_{0\lambda}>M_{J0,\lambda}$ for $\lambda\in\mathcal{C}_0$, and $M_{0\lambda}<M_{J0,\lambda}$ for $\lambda\in\mathcal{C}_4$ (equalities here would mean no spinors lie in $\mathcal{P}_L$, hence would contradict $J\notin\mathcal{J}_L(M)$). Choosing $\lambda=\lambda^{2\sqrt{r}}$ here then forces $d=\sqrt{r}$, hence $\ell=1$. Since $J_s\in\mathcal{J}_{L,R}(\mathcal{D}(\sqrt{r},1|\sqrt{r},1))$, we know $J_s\in\mathcal{J}_{L,R}(M)$, provided we conjugate M appropriately. One consequence of this is that 16 must divide r, since if $4\|r$, then $\mathcal{C}_1\cap\mathcal{P}_s=\emptyset$. Another consequence is that $\Lambda_{r-1}\notin\mathcal{P}_{L,R}(M)$, and so $\mathcal{M}_L(\Lambda_r)$ must be positive.

Now consider those M with $M_{0,J0}=1$. For $\lambda\in\mathcal{C}_0$, we know $\mathcal{M}_L(\Lambda_r)=M_{0\lambda}-M_{J0,\lambda}$ must be positive and independent of λ. Since $M_{0\lambda}+M_{J0,\lambda}=2$, the only possibility is that $M_{0\lambda}=\mathcal{M}_L(\Lambda_r)=2$ and $M_{J0,\lambda}=0$. Similarly, for $\lambda\in\mathcal{C}_4$ we find $M_{J0,\lambda}=2$ and $M_{0\lambda}=0$. This determines all entries of M, and we obtain $\mathcal{D}^{ii}$.

When $M_{0,J0}=0$ the equality $M\,\mathcal{I}[J]=\mathcal{D}^{ii}$ fixes most entries of M. $M_{J^i\Lambda_r,J^i\Lambda_r}=1$ is forced by evaluating $MS=SM$ at $(J^i\Lambda_r,0)$. We thus obtain $\mathcal{D}^i$.

5. The level 3 physical invariant classification

Write $0(M)$ for the set of all weights coupled to 0 (i.e. all weights λ obeying either $M_{0\lambda}\neq0$ or $M_{\lambda0}\neq0$). We want to show that $0(M)\subset\mathcal{S}_{sc}0$ for any physical invariant M of $B_{r,3}$ or $D_{r,3}$ (except for the $D_{7,3}$ exceptionals).

Consider first $B_{r,3}$. Write $n=2r+2$, $\gamma^i\stackrel{\rm def}{=}\Lambda_i$ for $i<r$, $\gamma^r\stackrel{\rm def}{=}2\Lambda_r$, $\mu^i\stackrel{\rm def}{=}\Lambda_i+\Lambda_r$ for $1\le i<r$, and $\mu^r\stackrel{\rm def}{=}3\Lambda_r$. There are $3r+4$ weights in P_+: 0, $3\Lambda_1=J_b0$, Λ_r, $2\Lambda_1+\Lambda_r=J_b\Lambda_r$, γ^i, $J_b\gamma^i$, and μ^i. The norms are

$$(J_b^i\Lambda_r+\rho)^2=\rho^2+\frac{r^2}{2}+\frac{r}{4}+2ni \tag{5.1a}$$

$$(J_b^i\gamma^a+\rho)^2=\rho^2+a\,(2r+1-a)+ni \tag{5.1b}$$

$$(\mu^a+\rho)^2=\rho^2+\frac{r}{4}(2r+1)+a\,(n-a)\ , \tag{5.1c}$$

for $i\in\{0,1\}$ and $1\le a\le r$, while the q-dimensions are given by

$$\mathcal{D}(\gamma^a)=\frac{\sin(\pi\,(2a+1)/2n)}{\sin(\pi/2n)} \tag{5.2a}$$

$$\mathcal{D}(\mu^a)=\sqrt{2}\,\frac{\sin(\pi\,(r+1-a)/n)}{\sin(\pi/2n)}\ . \tag{5.2b}$$

The main use of q-dimensions here is for reading off the Galois parities: $\epsilon_\sigma(\lambda)\,\epsilon_\sigma(0) = \mathrm{sgn}[\sigma\mathcal{D}(\lambda)]$ (see (3.2)). The automorphisms σ_ℓ are parametrised by integers $0 < \ell < 4n$ coprime to n. For example, $\sigma_\ell\sqrt{2} = \sqrt{2}$ if $\ell \equiv \pm 1 \pmod 8$, otherwise it equals $-\sqrt{2}$.

The first step is to show that no spinor λ (i.e. $\lambda \in P_+$ with λ_r odd) can couple to 0. Equation (5.1a) tells us $\Lambda_r, J_b\Lambda_r \not\in 0(M)$. Suppose for contradiction that $\mu^a \in 0(M)$ for a odd. Equation (5.1c) says 4 divides r. Take $\ell = n-1$: then (5.2b) says $\epsilon_\sigma(\mu^a)\,\epsilon_\sigma(0) = -1$, which contradicts the Galois selection rule (3.2c). Next, suppose instead $\mu^a \in 0(M)$ for a even. Equation (5.1c) says 8 divides r. Then checking each of the 4 cases ($\frac{a}{2}$ even/odd; $a \le \frac{r}{2}$ or $a > \frac{r}{2}$), we find that one of $\ell = \frac{n}{2} \pm 2$ will violate the Galois selection rule.

Thus the only possible weights coupling to 0 are in $\mathcal{J}_b\gamma^a$ and $\mathcal{J}_b0$. The Galois selection rule (3.2c) for them reduces precisely to that of $A_{1,4r+2}$, which was solved in Lemma 5 of [6]. Incidently, the proof of that lemma could have been simplified *enormously* by rewriting the parity trigonometrically:

$$\epsilon'_\ell(a\Lambda'_1)\,\epsilon'_\ell(0') = \mathrm{sgn}(\cos(\pi\ell a/m) - \cos(\pi\ell\,(a+2)/m))$$

where primes denote the quantities in $A_{1,m-2}$. Anyways, what we find is that for $r \ge 3$, no γ^a can satisfy the Galois selection rule (there is however a solution for $r = 2$, corresponding to the $C_{2,3}$ exceptional). Thus every physical invariant for $B_{r,3}$ is an $\mathcal{ADE}_7$-type invariant.

Moreover, $(J_b0 + \rho)^2 \equiv \rho^2 + n \pmod{2n}$, and hence any $\mathcal{ADE}_7$-type invariant will be an automorphism invariant. These were classified in [11], and we find that the only ones for $B_{r,3}$ are $M = I$ and $M = \mathcal{I}[J_b]$.

Incidently, it is particularly easy to classify the $B_{r,3}$ automorphism invariants: q-dimensions and (3.5b) tell us that $\pi\lambda \in \mathcal{J}_b\lambda$, and then use fusion coefficients or the values $S_{\Lambda_1\lambda}$ to show that $\pi\Lambda_r = \Lambda_r$ implies $M = I$, whereas $\pi\Lambda_r = J_b\Lambda_r$ implies $M = \mathcal{I}[J_b]$.

The proof for $D_{r,3}$ is easier. Put $n = 2r+1$. Then each of the $4r+8$ weights in P_+ can be mapped by a simple current J and possibly C_1 to one of $\gamma^0 \stackrel{\text{def}}{=} 0$, $\gamma^i \stackrel{\text{def}}{=} \Lambda_i$ for $1 \le i \le r-2$, $\gamma^{r-1} \stackrel{\text{def}}{=} \Lambda_{r-1} + \Lambda_r$, or $\gamma^r \stackrel{\text{def}}{=} 2\Lambda_r$. Norms and q-dimensions are given by

$$(\gamma^a + \rho)^2 = \rho^2 + a\,(2r - a) \tag{5.3a}$$

$$\mathcal{D}(\gamma^a) = \frac{\sin(\pi\,(2a+1)/2n)}{\sin(\pi/2n)} \tag{5.3b}$$

with a factor of $\frac{1}{2}$ on the right side of (5.3b) if $a = r$. The norms for $C_1^i J\gamma^a$ of course can now be obtained using (2.2b), while its q-dimension equals that of γ^a. Lemma 5 of [6] again applies, and we find that the only possibility for an anomolous coupling with 0 is $\mathcal{J}_d\gamma^5$ at $r = 7$. The usual arguments (see e.g. [7]) allow us to construct the exceptional physical invariant, and we obtain either $\mathcal{E}(D_{7,3})$ or $C_1\,\mathcal{E}(D_{7,3})$.

All other $D_{r,3}$ physical invariants will be $\mathcal{ADE}_7$-type invariants. The automorphism invariants were classified in [11]: we get $M = C_i$ and $C_i\,\mathcal{I}[J_v]$, as well as (for $r \equiv 4 \pmod 8$) the additional ones in $C_i\langle\mathcal{I}[J_s], \mathcal{I}[J_c]\rangle$, for arbitrary conjugation C_i.

The remaining $\mathcal{ADE}_7$-type invariants are found in [10], but we will provide an alternate argument here. Let M be any $\mathcal{ADE}_7$-type invariant which is not an automorphism invariant. Hitting M on either side if necessary with C_1, Lemma 3.1(b)(i) and (3.1) tell

us that it is sufficient to consider $\mathcal{J}_L(M)=\mathcal{J}_R(M)=\mathcal{J}_s$, with $8|r$. There are precisely $r+2$ orbits $\mathcal{P}_s/\mathcal{J}_s$: $\mathcal{J}_s 0$, $\mathcal{J}_s(J_v^a\gamma^a)$ for $1\le a\le r$, and $\mathcal{J}_s(2\Lambda_{r-1})$. Note that there are no J_s-fixed-points (since the level is odd) – this simplifies enormously the argument. This means (Lemma 3.1(c)) that there is a permutation π on those $r+2$ orbits such that (3.5) holds. In particular, the q-dimensions $\mathcal{D}(\lambda)$ and $\mathcal{D}(\pi\lambda)$ must be equal. Now (5.3b) tells us $\mathcal{D}(\gamma^0)<\mathcal{D}(\gamma^1)<\cdots<\mathcal{D}(\gamma^{r-1})$ and $\mathcal{D}(\gamma^r)=\mathcal{D}(\gamma^{\frac{r-1}{3}})$, and hence $\pi(\mathcal{J}_sJ_v^a\gamma^a)=\mathcal{J}_sJ_v^a\gamma^a$ for $a<r$ (to eliminate $\pi(\mathcal{J}_sJ_v\gamma^{\frac{r-1}{3}})=\mathcal{J}_sC_1^i\gamma^r$, use the fact that π must be a symmetry of the fusion $(J_v\Lambda_1)\boxtimes\Lambda_{\frac{r-1}{3}-1}=(J_v\Lambda_{\frac{r-1}{3}-2})\boxplus(J_v\Lambda_{\frac{r-1}{3}})\boxplus(\Lambda_{\frac{r-1}{3}-1})$). Then $M=\mathcal{I}[J_s]$ if π fixes $\mathcal{J}_s(2\Lambda_r)$, and $M=C_1\,\mathcal{I}[J_v]\,\mathcal{I}[J_s]$ if instead $\pi(\mathcal{J}_s(2\Lambda_r))=\mathcal{J}_s(2\Lambda_{r-1})$.

6. Concluding remarks.

Something very unusual happens for $B_r^{(1)}$ and $D_r^{(1)}$ at level 2, as has been noticed previously in the literature [4,11,16]. Indeed, many of the techniques available for generic algebras and levels – most significantly the Galois selection rule (3.2c) – break down at $B_{r,2}$ and $D_{r,2}$. This is a symptom of the existence here of a large family of exceptionals and is our motivation for doing their physical invariant classification. In this paper we also accomplish this classification for $B_{r,3}$ and $D_{r,3}$ – they follow quickly from a lemma solving the Galois selection rule for $A_1^{(1)}$. Thus this paper adds four more notches to the surprisingly barren bedpost representing families of $X_{r,k}$ for which the physical invariant classification has been completed. In the process we find infinitely many new exceptionals for both $B_{r,2}$ and $D_{r,2}$. The only level 3 exceptionals occur at $D_{7,3}$ and $B_{2,3}$ (which more properly should be written $C_{2,3}$).

Our explanation for the rich structure of level 2 physical invariants should be clear from the argument of section 4: rank-level duality relates $B_{r,2}$ and $D_{r,2}$ to $U_{1,2r+1}$ and $U_{1,2r}$, respectively, and $U_{1,n}$ has a known rich family of physical invariants [8]. An alternate explanation is offered in [16], using the $c=1$ orbifolds $SO(N)_1\times SO(N)_1/SO(N)_2$.

The only other low-level classifications, for any of the algebras, which are important are $B_{r,k}$ and $D_{r,k}$ for $k=4$ and 8, and to a lesser extent all other $k\le 6$ for these orthogonal algebras and the trivial case $A_{r,1}$. The reason again is rank-level duality: it breaks down or at least takes a different form for these algebras and levels. The reason the physical invariant classification for $B_{r,8}$ and $D_{r,8}$ would be interesting is that rank-level duality associates to it the very special algebra $D_4^{(1)}$, and D_4 triality is already known to give families of $B_{r,8}$ and $D_{r,8}$ exceptionals [18]. $B_{r,4}$ and $D_{r,4}$ are interesting because their Galois selection rules can be solved but have many solutions – this is usually a sign of exceptional chiral extensions (hence exceptional physical invariants). It should be possible to do these classifications with our current understanding.

The $C_{2,k}$ classification should be straightforward, and would also imply the $C_{r,2}$, $B_{r,5}$ and $D_{r,5}$ classifications. All $C_{2,k}$ physical invariants are known for $k\le 500$, and exceptionals appear only at $k=3,7,8,12$. Much more valuable, but more difficult, would be the $G_{2,k}$ classification. Its only exceptionals for $k\le 500$ appear at $k=3,4$. A very safe conjecture is that there are no new $C_{2,k}$ and $G_{2,k}$ exceptionals.

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